The Slope of A Line: It Is Not All Uphill From Here?

We have already talked about several parts to a linear equation, or linear function. We know that we can graph a linear function by plotting points and using the *x*-intercept and *y*-intercept. Now, we are going to look at another method of graphing using what is called slope. Let’s begin by defining slope and practice what it means to be the slope of a line before we start graphing.

We measure the slope of the line as a ratio of **vertical change** to **horizontal change**. Really, we are looking for the steepness of a line.

Slope is usually designated by the letter *m*.

$$\text{slope} = m = \frac{\text{change in } y \text{ (vertical change)}}{\text{change in } x \text{ (horizontal change)}} = \frac{\text{rise}}{\text{run}}$$

Here is the formula for slope = $\mathrm{m} = \frac{\mathrm{y}_2 - \mathrm{y}_1}{\mathrm{x}_2 - \mathrm{x}_1}$. Instead of memorizing the formula, let’s think about what it means. To find the **slope** of a line we take the difference in the *y*-coordinates divided by the difference in the *x*-coordinates.

<u>Example 1:</u> If we have the two points below:

$$(\underset{\underset{x_1}{\uparrow}}{1},\ \underset{\underset{y_1}{\uparrow}}{6})\quad(\underset{\underset{x_2}{\uparrow}}{7},\ \underset{\underset{y_2}{\uparrow}}{11})$$

We can identify the variables of the given ordered pairs as (x_1, y_1) and (x_2, y_2). Note that the subscripts are written so that we know that 1 is the *x*-coordinate from the first ordered pair and that 7 is the *x*-coordinate from the second ordered pair. Similarly, 6 is the *y*-coordinate from the first ordered pair and 11 is the *y*-coordinate from the second ordered pair.

Now, let’s take this in steps:

$m = \frac{y_2 - y_1}{x_2 - x_1}$ First, replace y_2 and y_1 with the given values.

$m = \frac{11-6}{x_2 - x_1}$ Now replace x_2 and x_1 with the given values.

$m = \frac{11-6}{7-1}$ To calculate the slope, take the difference in the numerator and the difference in the denominator.

$m = \frac{5}{6}$ Now you have the slope (or steepness) of the line between the two points (1, 6) and (7, 11).

You may be asking yourself, okay, “Now what does this mean?”. We can look at our result of $m = \frac{5}{6}$ in two ways. We could say that *y* changes +5 units when *x* changes +6 units. So, if we begin at the point (1, 6) on the coordinate plane, we can reach the point (7, 11) on the same line by moving up (positive) 5 units and then right (positive) 6 units.

What if we look at the slope, $m = \frac{5}{6}$ as $m = \frac{-5}{-6}$. We can do this since $m = \frac{5}{6} = \frac{-5}{-6}$. Then, we could say that *y* changes -5 units when *x* changes -6 units. So, if we begin at the point (7, 11) on the coordinate plane, we can reach the point (1, 6) on the same line by moving down (negative) 5 units and then left (negative) 6 units.

Examples:

Now, let’s do another example together. Write down the formula (just for practice) and find the slope of the line between the two points given. Then describe what the slope means in two ways.

2. $(-2,-5)$, $(3,-5)$

The formula for slope = $\frac{y_2 - y_1}{x_2 - x_1}$.

So, substituting the values given into the slope equation we have,

$$m = \frac{-5-(-5)}{3-(-2)}$$

$$m = \frac{-5+5}{3+2}$$

$$m = \frac{0}{5} = 0$$

Since the slope is $m = \frac{0}{5}$, we could say that the *y* value changes 0 units when *x* changes +5 units. Alternatively, slope could be $m = \frac{0}{-5}$, and we could say that the *y* value changes 0 units when *x* changes -5 units.

Here are a couple for you to try. Write down the formula (just for practice) and find the slope of the line between the two points given. Then describe what the slope means in two ways.

3. $(3,-2),\ (-1,-6)$

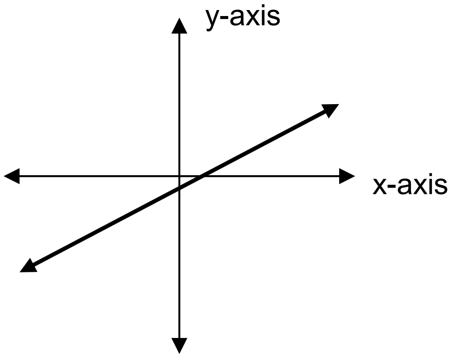
4. $(4,3), (-2, 5)$

Positive and Negative Slopes

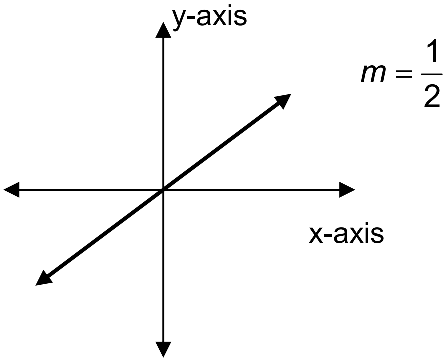
Linear functions can have positive and negative slopes.

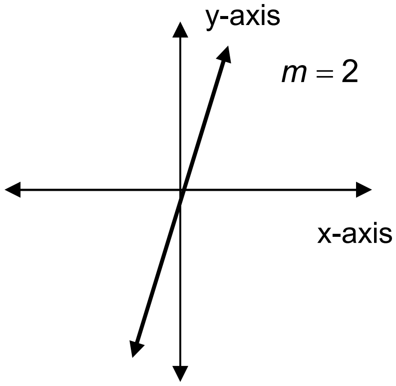
A line with a **positive slope** is called an increasing or a rising line.

If the line goes up when looking from left to right, the slope is positive.



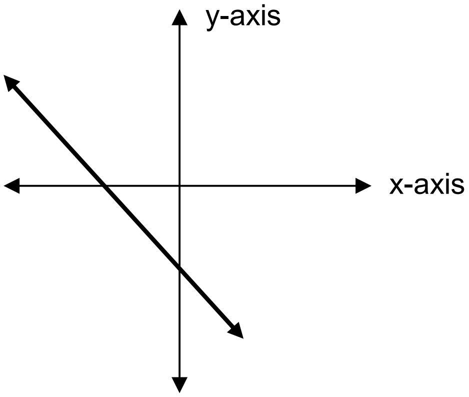
Notice in the graphs below that for a line with positive slope *m*, as the slope increases, the line becomes *steeper*.





A line with a **negative slope** is called a decreasing or descending line.

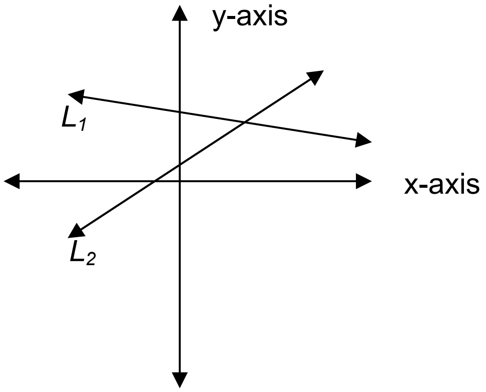
If the line goes down when looking from left to right, the slope is negative.



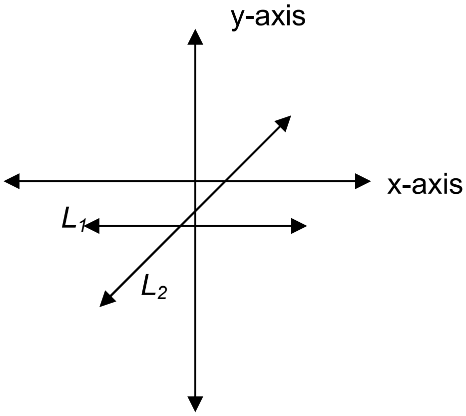
Examples:

Let's look at some interesting examples. Decide whether l_1 or l_2 has the greater slope.

5.



6.



L_1 has a negative slope and L_2 has a positive slope. Therefore, L_2 has the greater slope.

We are almost to the point where we can graph more easily. Before we do though, let's talk about a special form for an equation of a line.

One of the most useful forms of an equation of a line is called slope-intercept form. Being able to write an equation in this form will help you graph a linear function.

Slope-Intercept Form

When a linear equation in two variables is written in slope-intercept form,

$$y = mx + b$$

then *m* is the slope of the line and $(0, b)$ is the *y*-intercept of the line.

Examples:

To practice the parts of slope-intercept form let's go ahead and find the slope and *y*-intercept of each line. We will do the first one together.

7. $f(x) = -2x + 6$ Remember in function notation *y* is replaced by *f(x)*.

When an equation is written in slope-intercept form, the slope of the line is the coefficient of the variable *x*. For our equation, the slope is -2.

When an equation is written in slope-intercept form, the *y*-intercept of the line is the constant in the equation. For our equation, the *y*-intercept is (0, 6).

Now, you try a couple. Find the slope and *y*-intercept of each line.

8. $f(x) = 3x - 7$

9. $f(x) = -\frac{2}{3}x - \frac{1}{2}$

Slope ______ *y*-intercept __________

Slope ______ *y*-intercept ___________

Now, with this equation we need to do a little work before we can identify the slope and y-intercept

10. $2x - 3y = 10$ Begin by solving for y

$-3y = -2x + 10$ Subtract 2x from each side

$\frac{-3y}{-3} = \frac{-2x}{-3} + \frac{10}{-3}$ Divide each term on both sides of the equals by –3.

$y = \frac{2}{3}x - \frac{10}{3}$ Now, identify the slope and y-intercept.

Slope is $m = \frac{2}{3}$ *y*-intercept is $\left(0, \frac{-10}{3}\right)$

Okay, your turn. Find the slope and *y*-intercept of each line.

11. $3x + 4y = 0$

12. $-6x + 5y = 30$

Slope ______ *y*-intercept _________

Slope ______ *y*-intercept __________

Since we are talking about slope and the slope-intercept form of an equation of a line, let's look at parallel and perpendicular lines. We will find that the equations and graphs of parallel and perpendicular lines have special relationships.

Parallel and Perpendicular Lines

Parallel lines are two lines that never intersect. We know that two nonvertical lines are parallel if they have the same slope and different *y*-intercepts.

> Mathematically, if $l_1 : y = m_1x + b_1$ and $l_2 : y = m_2x + b_2$, then $m_1 = m_2$ when $b_1 \neq b_2$.

Perpendicular lines are two lines that intersect each other and form a right angle at their intersection. We know that two nonvertical lines are perpendicular if the product of their slopes is -1.

> Two nonvertical lines are perpendicular if the slope of one is the opposite reciprocal of the slope of the other.
>
> Mathematically, if $l_1 : y = m_1x + b_1$ and $l_2 : y = m_2x + b_2$, then $m_1m_2 = -1$ or $m_2 = -\frac{1}{m_1}$

So, in other words, if we want to decide if two lines are parallel, perpendicular, or neither; find the slope.

Examples:

In these examples we want to determine whether the lines are parallel, perpendicular, or neither. As always, we will do one together first.

13. $$\begin{aligned} f(x) &= -3x + 6 \\ g(x) &= 3x + 5 \end{aligned}$$

Since our equations are in slope-intercept form we can easily identify the slope of each function.

The slope of *f* is -3 and the slope of *g* is 3. Let's ask ourselves some questions:

a. Are the slopes the same? Does -3 = 3?

No way…so these lines are not parallel.

b. Are the slopes opposite and reciprocals?

Well, the slopes are opposites of each other, but they are not reciprocals. AND, when we multiply -3 times 3 we do NOT get -1.

So, these lines are not perpendicular.

Our conclusion is that the two lines may intersect each other, but are not parallel or perpendicular. We must answer neither.

Now, try this problem. We want to determine if the two lines are parallel, perpendicular, or neither. Notice that both lines are not in slope-intercept form. Here is your hint: solve each equation for *y* (so that the equation will be in slope-intercept form) and then compare the slope of each equation.

14. $$\begin{aligned} 2x - y &= -10 \\ 2x + 4y &= 2 \end{aligned}$$

Another useful application of the characteristics of parallel and perpendicular lines is that by knowing the slope of one line, we can determine the slope of another line that is either parallel or perpendicular to the original line.

Examples:

15. Find the slope of a line parallel to the line $f(x) = -\frac{3}{2}x + 4$?

Well, we know that parallel lines have the same slope. Since the equation given has a slope of $m = -\frac{3}{2}$, then a line that is parallel to the line given, $f(x) = -\frac{3}{2}x + 4$ would also have a slope of $m = -\frac{3}{2}$.

16. Find the slope of the line perpendicular to the line $5x - 2y = 6$. (Hint: Solve for *y*.)

In summary, to find the slope of a line we take the difference in the *y*-values divided by the difference in the *x*-values. When given an equation of a line, we can easily determine the slope and *y*-intercept by writing our equation in slope-intercept form. Writing linear equations in slope-intercept form will help us graph and determine if two lines are parallel, perpendicular, or neither.